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RESEARCH ARTICLE

Topology-Disturbing Objects: A New Class of 3D Optical Illusion

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A new class of objects, called topology-disturbing objects, is presented. These objects appear to disturb topological properties when they are seen from two specific viewpoints; for example, two objects appear to be separated when seen from one viewpoint while they appear to be intersecting when seen from the other viewpoint. This cannot happen physically, but due to the phenomenon of an optical illusion, it can be perceived to occur. A general method for designing this class of objects is presented and examples are given. These objects might provide a new resource for arts and entertainment.

Keywords: Ambiguous cylinder, anomalous object, impossible object, inconsistent mirror image, visual illusion.

AMS 2010 Mathematics Subject Classification: 00A66 (Mathematics and Visual Art, Visualization)

1. Introduction

The topology of a geometric object provides the most fundamental properties, in the sense that they are not disturbed under any continuous transformation; for example, connected components remain connected, and disconnected ones remain disconnected. Topological properties are thus preserved when objects are rotated around a vertical axis, because the rotation is one of the simplest continuous transformations. However, we found a class of objects for which the topological properties appear to be disturbed when they are seen from the two viewpoints. Of course, physically, this cannot happen, but an optical illusion can make humans perceive that it has happened.

An effective method for presenting two views simultaneously is to use a mirror. Placing an object and a mirror appropriately, we can see the object and its reflection in the mirror, and thus can compare the two appearances. If the topologies are different from each other, we might have a strong sense of wonder and impossibility. Thus, we can say that the topology-disturbing objects are those whose topologies appear to change in a mirror.

In the history of arts and entertainment, optical illusions have been used to create the impression of impossibility. Penrose and Penrose [10] and Escher [1, 5] drew pictures of impossible objects, which are optical illusions because they appear to represent three-dimensional (3D) structures, but ones that could not possibly exist. Moreover, these impossible objects have been realized as 3D structures by using optical illusions generated by tricks involving discontinuity and curved surfaces [4]. Physically impossible motion can also be created by antigravity slopes, where the orientations of the slopes are perceived to be the opposite to their true orientations; in these illusions, balls appear to defy gravity and roll uphill [14, 15]. Another example is Hughes' 3D painting method known as reverse perspective [19]. He painted pictures on a 3D surface in such a way that near objects are painted on a part of the surface that is farther from the viewer,

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and far objects are painted nearer to the viewer. The result is that we perceive unexpected motion when we move our head [3, 9]. A similar depth-reversal trick known as the hollow-face illusion is used in haunted mansions to create a visual impression that the gaze of a statue is following the viewer [6, 13]. Recently, Sugihara found ambiguous cylinders that appear drastically different when viewed from two specific viewpoints [16, 17]. The objects presented in this paper are variants of ambiguous cylinders. Specifically, the original ambiguous cylinders disturb geometry, while the present objects disturb topology.

The visual effect of topology change can be considered a variant of traditional anamorphosis. In anamorphosis, painting on a plane or on a 3D surface looks meaningless when seen from a general viewpoint, while it becomes meaningful when seen from a single specific viewpoint.

A typical example is Gregory's 3D realization of the Penrose impossible triangle. The Penrose impossible triangle is an anomalous picture of an imaginary 3D structure that is evoked in our brain when we see the picture but that cannot be constructed physically [10]. Gregory [6] created a 3D model of an open path composed of three rods that appears to close into the Penrose impossible triangle when seen from a special viewpoint. This is an example of anamorphosis because it looks like nothing in particular when seen from a general viewpoint, but looks like an impossible triangle when seen from a unique special viewpoint.

Note that traditional anamorphosis gives meaningful appearance when it is seen from one special viewpoint. The topology-disturbing objects, on the other hand, are accompanied with two specific viewpoints, from which they appear to be meaningful but drastically different, and thus create the sense of impossibility. From this aspect of their nature we might consider the topology-disturbing objects as a kind of multiple anamorphosis. Note that there are various other classes of multiple anamorphoses. One class is multiple-silhouette sculptures such as "Encore" (1976) by Shigeo Fukuda, which gives a silhouette of a pianist and a silhouette of a violinist when seen from two special viewpoints, and "1, 2, 3" by James Hopkins, which gives three silhouettes of "1", "2", and "3". Another class is multiple-appearance wire frame art such as the one that appears to be an elephant and giraffes when seen from two special viewpoints.

We will show examples of topology-disturbing objects (Section 2), briefly review the principles of ambiguous cylinders (Section 3), and apply these principles to the design of topology-disturbing objects (Section 4). Next, we summarize a general condition for the constructability of the topology-disturbing object (Section 5), and give concluding remarks (Section 6). We also present a diagram that shows the unfolded surfaces of a simple example of a topology-disturbing object; from this, one may construct the object by paper crafting (Appendix A). Videos of topology-disturbing objects can also be found on YouTube [18].

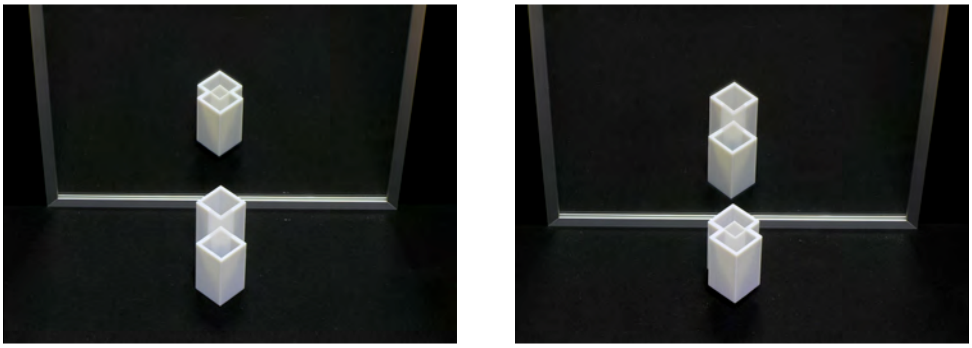
2. Examples of Topology-Disturbing Objects

Fig. 1(a) shows an example of a topology-disturbing object. The direct view of this object consists of two rectangular cylinders that are separated from each other. A plane mirror is positioned vertically behind the object, and the object can also be seen in the mirror. However, in the mirror, the two cylinders appear to intersect each other. Thus, the direct view and the mirror image have different topologies, which seems impossible.

The mirror is an ordinary plane mirror, and hence it just gives us another view of the object. Therefore, a topology-disturbing object appears to have two different topological structures when viewed from two special directions. Actually, if we rotate the object around a vertical axis by 180 degrees, the direct appearance and the mirror image are interchanged, as shown in Fig. 1(b). This behavior is typical of topology-disturbing objects.

If we replace the rectangular cylinders with circular cylinders, we can construct a similar topology-disturbing object, as shown in Fig. 2. The direct view consists of two circular cylinders that are separated, but in the mirror view, they intersect.

Fig. 3 shows what happens if another cylinder is added, and the three cylinders are placed along a slanted line. As before, the three cylinders are disconnected in the direct view, while in



(a) (b)

Figure 1. Topology-disturbing pair of rectangular cylinders.

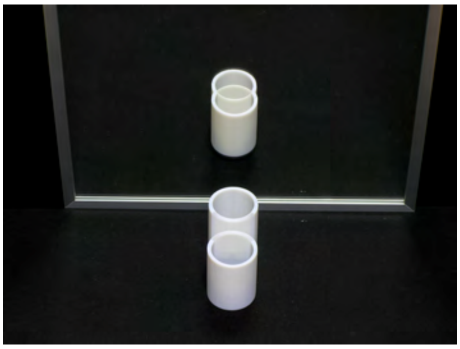


Figure 2. Topology-disturbing pair of circular cylinders.

the mirror view, they appear to intersect.

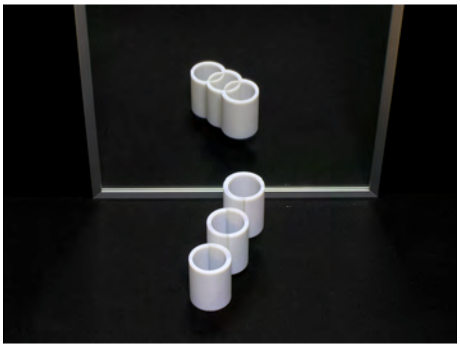


Figure 3. Topology-disturbing triplet of cylinders.

Fig. 4 shows another type of a topology-disturbing object. In the direct view, the two cylinders are concentric (nested). However, in the mirror, the cylinders appear to have changed shape and to intersect each other.

Fig. 5 shows another object, which consists of a cylinder and a plane. In the direct view, the plane passes through the cylinder, while in the mirror, the plane and cylinder are separated.

The above example are typical topology-disturbing objects. Each object consists of two or more cylinders (Fig. 5 includes a plane). In one view, they are disconnected, while in the other view, they intersect. Disconnection and intersection are topologically invariant properties, but in this case, they are not preserved in the mirror image, and thus we call them topology-disturbing objects.

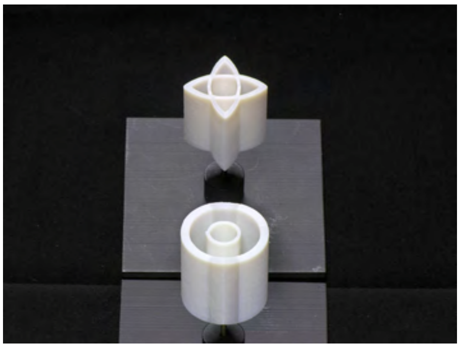


Figure 4. Topology-disturbing nesting cylinders.

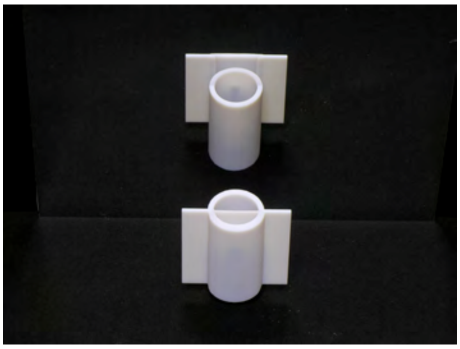


Figure 5. Topology-disturbing cylinder and a plane.

3. Principle of Ambiguous Cylinders

Topology-disturbing objects can be constructed by applying a variation of the method used to design ambiguous cylinders. Hence, in preparation, we will briefly review the principles of ambiguous cylinders [16]. An ambiguous cylinder is a cylindrical object that appears to have different structures when viewed from two special directions. They can be constructed in the following way.

As shown in Fig. 6, we fix an xyz Cartesian coordinate system so that the xy plane is horizontal, and the positive z direction orients upward. Let $\boldsymbol{v}_1 = (0, \cos\theta, -\sin\theta)$ and $\boldsymbol{v}_2 = (0, -\cos\theta, -\sin\theta)$ be two viewing directions; they are parallel to the yz plane, and they are directed downward at the same angle θ, but in opposite directions.

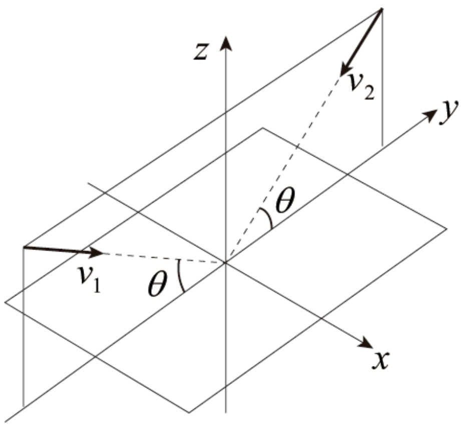


Figure 6. Two viewing directions parallel to the yz-plane.

Next, as shown in Fig. 7, on the xy plane, we fix two curves $a(x)$ and $b(x)$ for $x_0 \leq x \leq x_1$. Note that the initial points $a(x_0)$ and $b(x_0)$ have the same x coordinate, and similarly, the end points $a(x_1)$ and $b(x_1)$ have the same x coordinate. Moreover, these two curves are x-monotone. For each x, $x_0 \leq x \leq x_1$, we consider the line passing through $a(x)$ and parallel to $\boldsymbol{v}_1$, and the line passing through $b(x)$ and parallel to $\boldsymbol{v}_2$. These two lines are included in the same plane parallel to the yz plane, and hence they have a point of intersection. Let this point be denoted by $c(x)$. Then, $c(x)$, $x_0 \leq x \leq x_1$, forms a space curve. The curve $c(x)$ coincides with $a(x)$ when it is seen in the direction $\boldsymbol{v}_1$, and it coincides with $b(x)$ when it is seen in the direction $\boldsymbol{v}_2$.

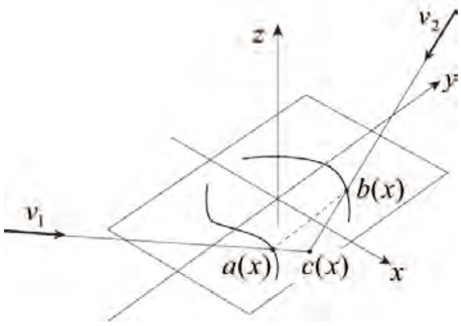


Figure 7. Space curve whose appearances coincide with two given plane curves.

Finally, we choose a vertical line segment L and move it in such a way that it remains vertical, and the upper terminal point traces along the curve $c(x)$, $x_0 \leq x \leq x_1$. Let S be the surface swept by L. S is a surface with vertical rulers, and the vertical length of S is the same as the length of L. Therefore, when viewed, it appears to be a cylindrical surface with a constant height. In other words, we are likely to perceive it as a cylindrical surface whose upper and lower edges are obtained by cutting the surface with a plane perpendicular to the axis of the cylinder. This may be the result of the preference for rectangularity in the human vision system [11, 12]. As a result, the upper edge appears to be the plane curve $a(x)$ when seen in the direction $\boldsymbol{v}_1$, and appears to be $b(x)$ when seen in the direction $\boldsymbol{v}_2$. This method can be used to construct an ambiguous cylinder that has the two desired appearances $a(x)$ and $b(x)$ when it is seen from the special viewing directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively.

The surface thus constructed is x-monotone and hence is not closed. If we want a closed cylinder, we can apply the above method twice, once for the upper half of the cylinder and once more for the lower half.

4. How to Make Topology-Disturbing Objects

We have reviewed a method for constructing a cylindrical surface whose upper edge has two desired appearances when it is viewed from two special directions. We can use this method to construct topology-disturbing objects, in the following way.

Consider the object shown in Fig. 1; Fig. 8 shows the shape of the sections of the object that we see. From the direct view, we perceive two nonintersecting rectangles as shown in (a), and from the mirror image, we perceive two intersecting rectangles as shown in (b). We decompose the shape in (b) into two nonintersecting closed curves as shown in (c), where, for simplicity, the two closed curves are displaced so that they do not touch. We apply the method for ambiguous cylinders to the curves in (a) and (c). That is, we construct two ambiguous cylinders, one for the upper pair and one for the lower pair in Fig. 8(a) and (c). Thus we obtain two cylinders, each of which has the desired appearance.

The remaining problem is to combine the two cylinders so that they appear to be separated when viewed from the first direction $\boldsymbol{v}_1$, and they appear to touch when viewed from the second

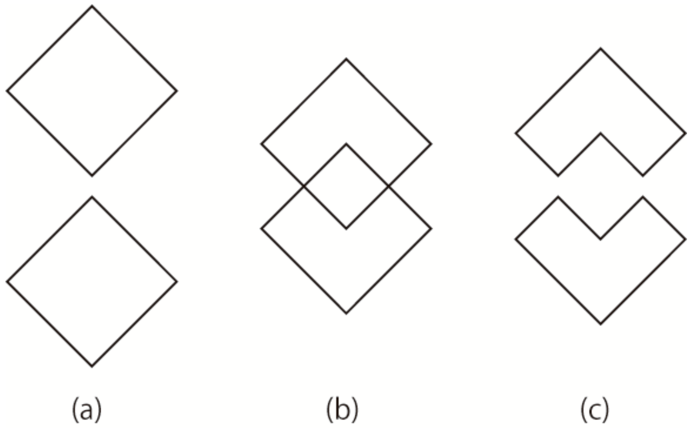


Figure 8. Desired appearance of two rectangular cylinders: (a) direct view; (b) view in the mirror; (c) decomposition of (b) into two nonintersecting shapes.

view direction $\boldsymbol{v}_2$; note that if we move the curves in Fig. 8(c) so that they touch, they will appear as in (b), which will be perceived as intersecting. For this purpose, we place the two cylinders in different vertical positions as shown in Fig. 9. The two cylinders are placed apart, and then their vertical positions are adjusted so that they appear to touch when seen from one of the viewing directions, as shown by the broken line in the figure. We will discuss later how to place cylinders in more details. From this method, we obtain a topology-disturbing object.

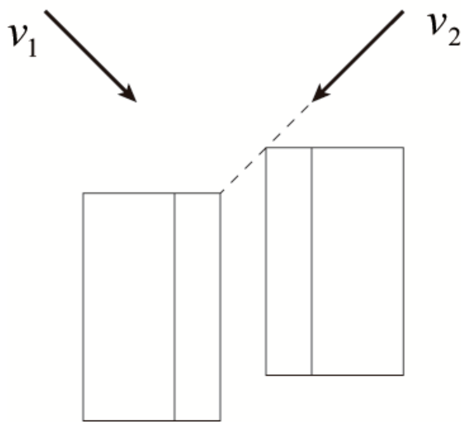


Figure 9. Adjustment of the heights so that they appear disconnected when viewed from one direction, but they appear to intersect when viewed from the other view direction.

A general view of the object shown in Fig. 1 is shown in Fig. 10. As seen in this figure, the two cylinders are fixed at different heights; they are connected by additional material that is invisible from either of the two special viewing directions.

Note that a single image does not have depth information, and hence its interpretation as a 3D object is not unique. However, when we see the object and its mirror image from a special viewpoint shown in Fig. 1, we usually perceive two separate cylinders and two intersecting cylinders. This perceptual phenomenon may be based on human preference and familiarity for canonical shapes such as a circle and a square. Strength of similar preference has been observed and studied in many contexts, including figure-ground discrimination [8], depth perception [2], visual search [20], and line drawing interpretation [7]. This is an important psychological aspect of the topology-disturbing objects, but we postpone this issue for future work. The main issue of the present paper is to point out the geometric feasibility of this class of illusory objects.

The same approach can be used for the other examples of topology-disturbing objects that were presented in Section 2.

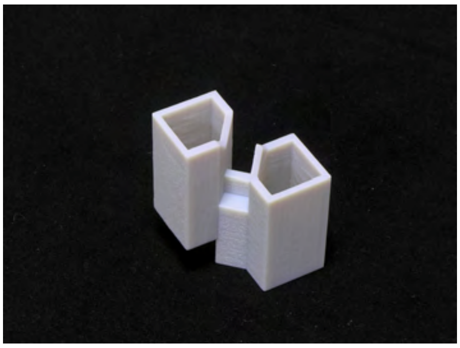


Figure 10. General view of the object shown in Fig. 1.

Fig. 11 shows the perceived shapes of the object presented in Fig. 2; Fig. 11(a) shows the direct view, in which the two cylinders are separated, and Fig. 11(b) shows the mirror image, in which the two cylinders intersect. We decompose the shape in (b) into two nonintersecting curves as shown in (c), and we then apply the above method to the curves in (a) and (c) to obtain two ambiguous cylinders. Finally, we adjust the distance between them and their height in order to obtain the object shown in Fig. 2. Fig. 12 shows a general view of the resulting object.

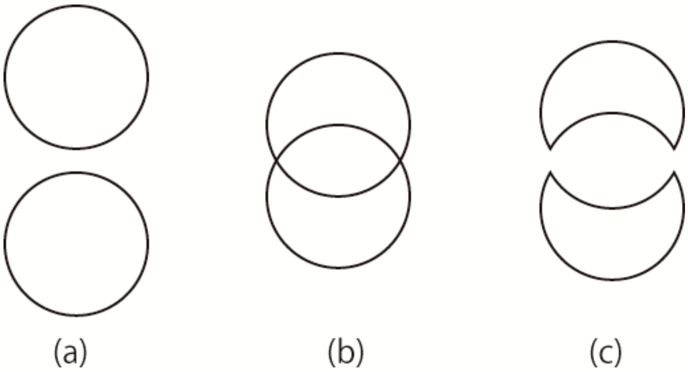


Figure 11. Perceived shapes of the object in Fig. 2.

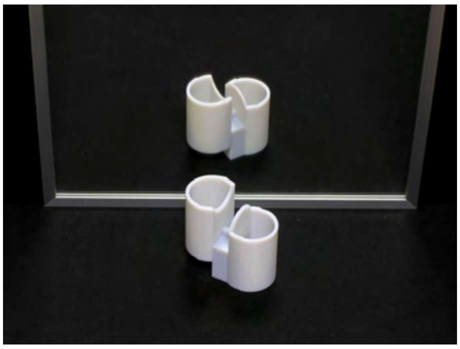


Figure 12. General view of the object shown in Fig. 2.

Fig. 13 shows the perceived shape of the object shown in Fig. 3. Fig. 13(a) is the direct view, and (b) is the mirror image. We decompose the shape in (b) into three nonintersecting curves, as shown in (c), and we then apply our method. A general view of the resulting object is shown in Fig. 14.

For the object shown in Fig. 4, we obtain the perceived shapes that are shown in Fig. 15; (a) shows the direct view, (b) shows the mirror image, and (c) shows the decomposed nonintersecting

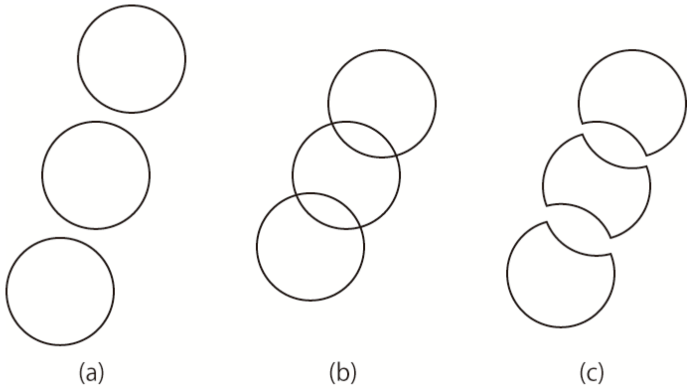


Figure 13. Perceived shapes of the object shown in Fig. 3.

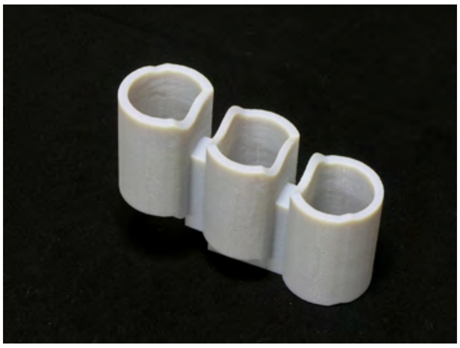


Figure 14. General view of the object shown in Fig. 3.

curves, where the inner curve is shrunk slightly in order to clarify that they do not intersect. We apply our method to the images in (a) and (c) to obtain the object shown in Fig. 4. A general view of that object is shown in Fig. 16.

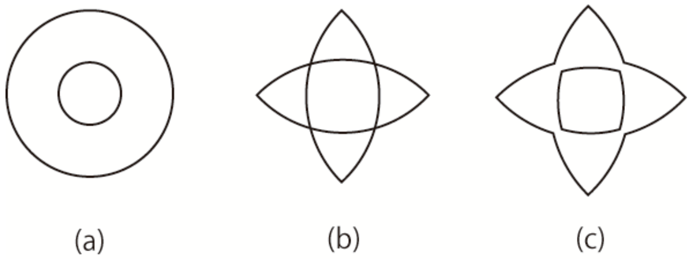


Figure 15. Perceived shapes of the object shown in Fig. 4.

For the object shown in Fig. 5, we obtain the shape diagram shown in Fig. 17; (a) shows the direct view, (b) shows the decomposed pair of nonintersecting curves, and (c) shows the mirror image. If we apply our method to (b) and (c), we obtain the object shown in Fig. 5. A general view of the object is shown in Fig. 18.

Three other examples of topology-disturbing objects are shown in Figs. 19, 20, and 21. In each figure, (a) shows the direct view and its image in a mirror, and (b) shows a general view of the object.

Fig. 19 shows an object composed of five cylinders. In the direct view, the three cylinders in the front row are touching, and the two cylinders in the back row are touching, but the two rows are separated. However, in the mirror image, all five cylinders intersect the adjacent cylinders.

Fig. 20 shows an object composed of a cylinder and two parallel planes. In the direct view,

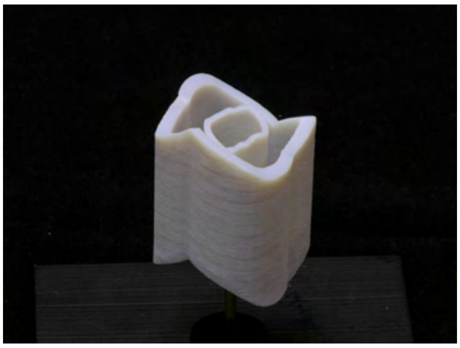


Figure 16. General view of the object shown in Fig. 4.

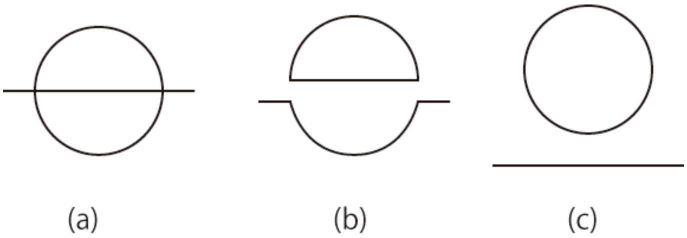


Figure 17. Perceived shapes of the object shown in Fig. 5.

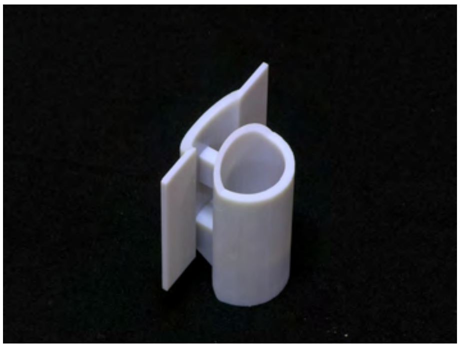


Figure 18. General view of the object shown in Fig. 5.

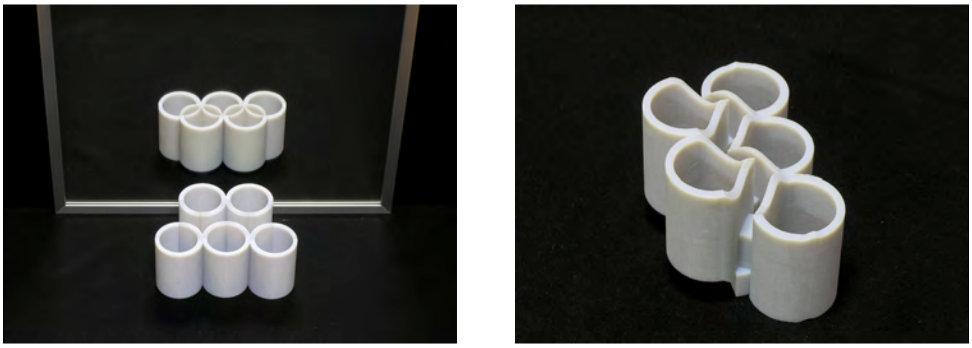
both of the planes cut through the cylinder, but in the mirror view, they are on opposite sides and separated from the cylinder.

Fig. 21 shows another object. In the direct view, it consists of two intersecting lens-shaped cylinders, but in the image they change to two pairs of touching but nonintersecting circular cylinders.

5. Constructability Condition

We have seen the construction process of topology-disturbing objects through examples. Next we consider a general condition under which a real 3D object can be constructed from a given pair of appearances.

Let F be a line drawing composed of curves drawn on the xy plane, where each curve is a non-self-intersecting closed curve such as a circle/rectangle or an open curve such as a line segment. We assume that each curve in F is not self-intersecting while different curves may intersect. We decompose the curves in F into x-monotone segments and represent them by equations $y = f_1(x), y = f_2(x), \ldots, y = f_n(x)$ in such a way that

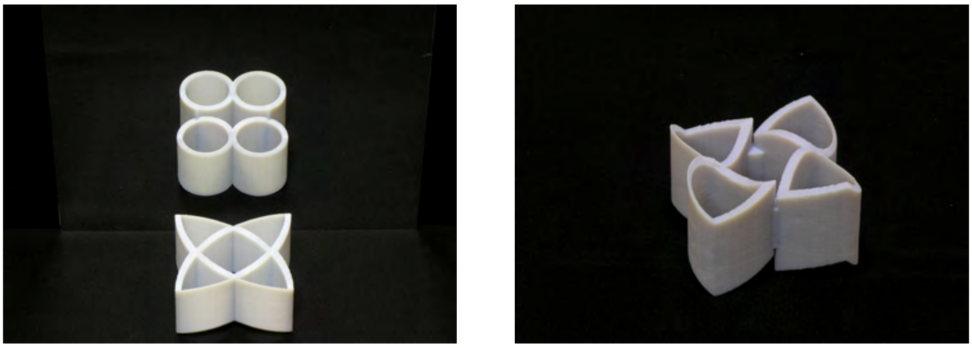


(a) (b)

Figure 19. Five cylinders with disturbed topology.

(a) (b)

Figure 20. Two parallel planes and a cylinder with disturbed topology.



(a) (b)

Figure 21. Four cylinders with disturbed topology.

(1) two segments do not cross each other (although they may touch), and
(2) an upper segment has a smaller segment number than the lower segment, i.e., $f_i(x) \geq f_j(x)$ implies $i < j$.

For example, suppose that F is the line drawing composed of two circles and a line segment shown in Fig. 22(a). This drawing can be decomposed into five x-monotone segments $f_1(x), f_2(x), \ldots, f_5(x)$ as shown in (b), where we slightly displaced the horizontal positions of curves so that the touching segments are separated for the convenience of understanding.

Similarly, let G be another line drawing obtained from F by replacing each curve in the direction parallel to the y axis, and $y = g_1(x), y = g_2(x), \ldots, y = g_m(x)$ are x-monotone segments obtained by decomposing G satisfying (1) and (2). We are interested in whether we can construct a cylindrical object whose appearances from two special viewpoints coincide with the drawings

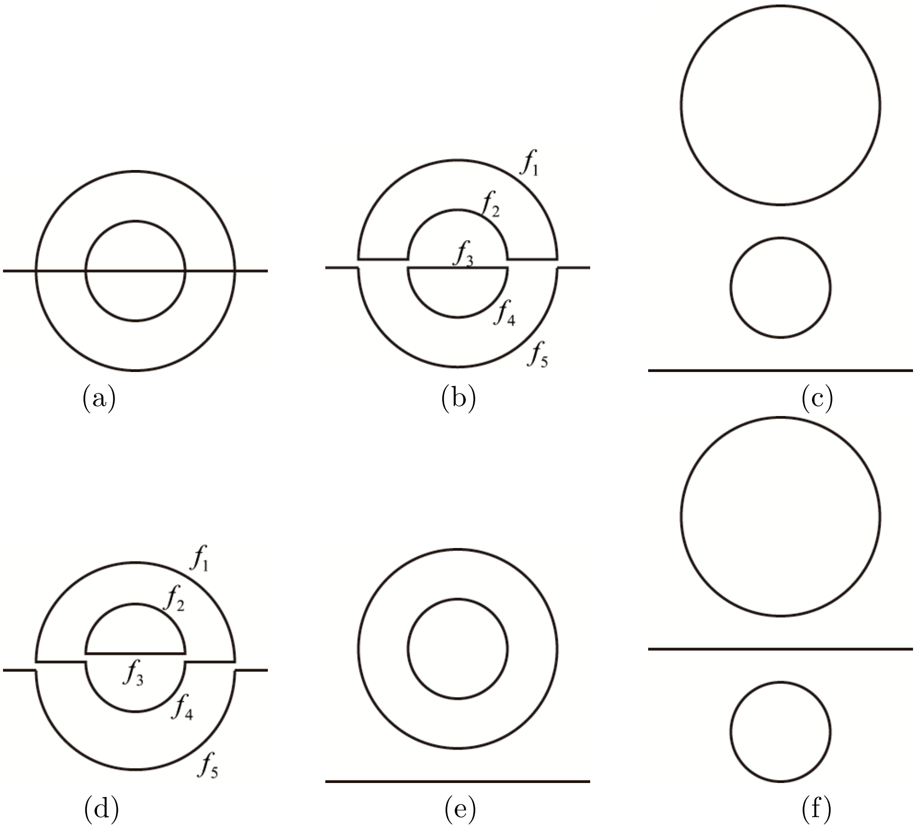


Figure 22. Line drawing and its decomposition into x-monotone segments: (a) first line drawing F; (b) decomposition of F; (c) second line drawing, which is consistent with (b); (d) another decomposition of F; (e) third line drawing, which is consistent with (d); (f) fourth line drawing, which is not consistent with F.

F and G. We can prove the next theorem.

Theorem We can construct a topology-disturbing object whose appearances coincide with the drawings F and G if the following conditions are satisfied:
(3) $m = n$, and
(4) for each i, $f_i(x)$ and $g_i(x)$ span the same x range; i.e., their leftmost points have the same x coordinate, and their rightmost points also have the same x coordinate.

A rough sketch of the proof is as follows. Suppose that the conditions (3) and (4) are satisfied. Then, we can construct the ambiguous cylinders corresponding to $f_i(x)$ and $g_i(x)$ for $i = 1, 2, \ldots, n$. Let the resulting ambiguous cylinders be $H_i, i = 1, 2, \ldots, n$. Next, place the vertical cylinders $H_1, H_2, \ldots, H_n$ so that their appearance coincides with the drawing F when they are seen along the first view direction. This is always possible, because the curves $f_1(x), f_2(x), \ldots, f_n(x)$ do not cross each other (see condition (1)). If we see this collection of the cylinders along the second view direction, each cylinder has the desired appearance specified by the second line drawing G, but their mutual positions may not coincide with G. Therefore, as a final step, we translate the cylinders in the direction parallel to the first view direction so that the relative positions in the second appearance coincide with G. Note that the appearance in the first view direction is not changed by these translations because they are in that direction. Note also that these translations can be done without collision because their order in the y direction is the same (see condition (2)). Thus, the theorem can be proved.

Let us go back to the example in Fig. 22. Suppose that the second drawing G is given as in (c). This drawing can be decomposed into five x-monotone segments by cutting the circles at the leftmost and the rightmost points. This decomposition of G together with the decomposition of F into (b) satisfies the conditions of the theorem, and hence we can construct the associated topology-disturbing object.

The decomposition of the drawing Fig. 22(a) is not unique. Another decomposition is shown in (d). This decomposition and the unique decomposition of the line drawing in (e) satisfy the conditions in the theorem, and hence we can construct a topology-disturbing object corresponding to the drawings (a) and (b).

Fig. 22 (f) shows still another drawing composed of the same two circles and the line segment. However, we cannot decompose the drawing (a) so that it is consistent with (f) in terms of condition (4). Hence, we cannot construct a topology-disturbing object associated with the drawings (a) and (f).

6. Concluding Remarks

We have presented a class of illusory objects, called topology-disturbing objects, which, when viewed from two special directions, appear to have different topologies; this gives us the impression that they are physically impossible. Therefore, this can be regarded as a new class of "impossible objects". Two viewing directions can be realized simultaneously by using a mirror, and hence, these objects can be displayed effectively (such as for an exhibition) by using a mirror.

These objects may be used in many contexts. In vision science, they offer new material for research on seeing; for example, in order to understand human vision, we must clarity why we easily perceive topological inconsistencies. In science education, these objects could be used to stimulate children to start thinking about visual perception. At least upon one's first encounter with topology-disturbing objects, they are surprising and mysterious, and thus their unusual visual effects might be used in the arts and entertainment. One aspect of our future work is to investigate the possibilities in these directions.

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Appendix A. Unfolded Surfaces of a Topology-Disturbing Object

The topology-disturbing objects shown in this paper were made by a 3D printer. Since they are in general composed of complicated curved surfaces, it is difficult to make them by hand. However, the object shown in Fig. 1 is an exception. It consists of planar faces, and hence, we can use paper crafting to construct it.

Fig. A.1 shows a diagram of the unfolded surfaces of the object. It consists of three components: A, B, and C. Components A and B correspond to the two rectangular cylinders, and component C is used to connect them. The light-gray areas indicate where they should be glued together. For components A and B, both the front and back sides are shown. Copy them onto stiff paper, cut out, and fold as indicated (solid interior lines). The white areas indicated by “a” and “b” are glued to the matching sections of C. The surfaces are then painted so that one rectangular cylinder appears to be blue and the other yellow.

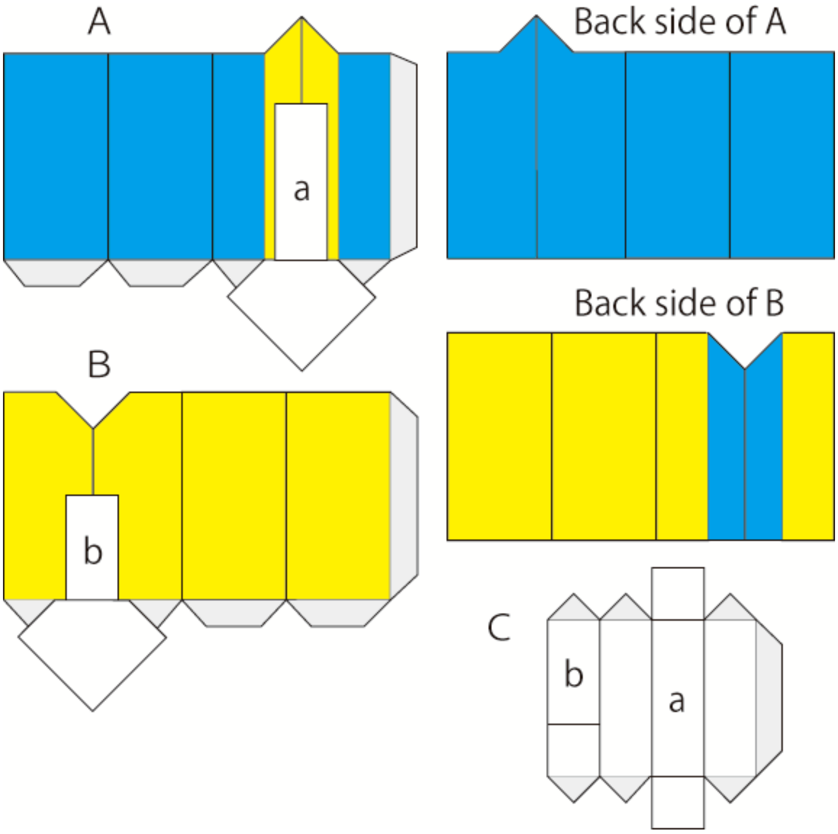

Figure A.1. Unfolded surfaces of the components of the object shown in Fig. 1.

Fig. A.2 shows the top view (upper part) and the side view (lower part) of the constructed object. As shown in this figure, component B should be placed slightly higher than component A. The viewing angle for this object is 45 degrees; that is, if we view this object from above at an angle of 45 degrees from the horizontal, as shown by $\boldsymbol{v}_1$ in Fig. A.2, the corners of the two cylinders will coincide and thus appear to intersect. If viewed from the opposite side, $\boldsymbol{v}_2$, the two cylinders appear to be disconnected, because they are at different heights.

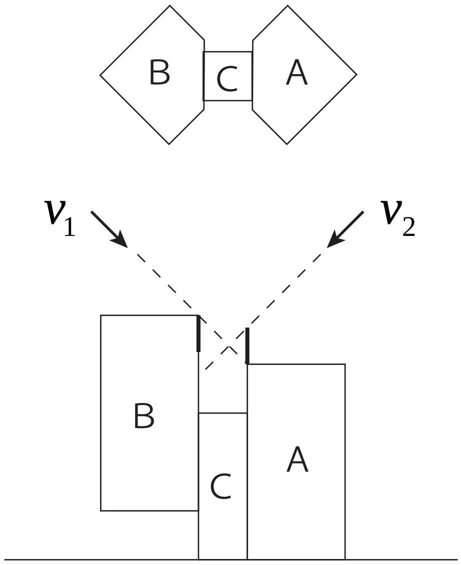

Figure A.2. Top and side views of the object.

We encourage the reader to construct this object for themself in order to better understand the actual shape of the object and the way in which we perceive this illusion.